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HOMOMORPHISMS OF INFINITELY GENERATED ANALYTIC SHEAVES

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Abstract. We prove that every homomorphism $\mathcal{O}_\zeta^E \to \mathcal{O}_\zeta^F$, with E and F Banach spaces and $\zeta \in \mathbb{C}^m$, is induced by a $\mathrm{Hom}(E, F)$-valued holomorphic germ, provided that $1 \le m < \infty$. A similar structure theorem is obtained for the homomorphisms of type $\mathcal{O}_\zeta^E \to \mathcal{S}_\zeta$, where $\mathcal{S}_\zeta$ is a stalk of a coherent sheaf of positive $\mathfrak{m}_\zeta$-depth. We later extend these results to sheaf homomorphisms, obtaining a condition on coherent sheaves which guarantees the sheaf to be equipped with a unique analytic structure in the sense of Lempert-Patyi.

1. Introduction

The theory of coherent sheaves is one of the deeper and most developed subjects in complex analysis and geometry, see [GR84]. Coherent sheaves are locally finitely generated. However, a number of problems even in finite dimensional geometry leads to sheaves that are not finitely generated over the structure sheaf $\mathcal{O}$, such as the sheaf of holomorphic germs valued in a Banach space; and in infinite dimensional problems infinitely generated sheaves are the rule rather than the exception. This paper is motivated by [LP07], that introduced and studied the class of so called cohesive sheaves over Banach spaces; but here we shall almost exclusively deal with sheaves over $\mathbb{C}^m$. In a nutshell, we show that $\mathcal{O}$-homomorphisms among certain sheaves of $\mathcal{O}$-modules have strong continuity properties, and in fact arise by a simple construction.

We will consider two types of sheaves. The first type consists of coherent sheaves. The other consists of plain sheaves; these are the sheaves $\mathcal{O}^E$ of holomorphic germs valued in some fixed complex Banach space E. The base of the sheaves is $\mathbb{C}^m$ or an open $\Omega \subset \mathbb{C}^m$. Thus $\mathcal{O}^E$ is a (sheaf of) $\mathcal{O}$-module(s). We denote the Banach space of continuous linear operators between Banach space E and F by $\mathrm{Hom}(E, F)$. Any holomorphic map $\Phi : \Omega \to \mathrm{Hom}(E, F)$ induces an $\mathcal{O}$-homomorphism $\phi : \mathcal{O}^E \to \mathcal{O}^F$. If $U \subset \Omega$ is open, $\zeta \in U$ and a holomorphic $e : U \to E$ represents a germ $e_\zeta \in \mathcal{O}_\zeta^E$, then $\phi(e) \in \mathcal{O}_\zeta^F$ is defined as the germ of the function $U \ni z \mapsto \Phi(z)e(z) \in E$. Following [LP07], such homomorphisms will be called plain. In fact, if Φ is holomorphic only on some neighborhood of ζ it still defines a homomorphism $\mathcal{O}_\zeta^E \to \mathcal{O}_\zeta^F$ of the local modules over the local ring $\mathcal{O}_\zeta$. Again such homomorphisms will be called plain.

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The first question we address is a how restrictive it is for a homomorphism to be plain. It turns out it is not restrictive at all, provided $0 < m < \infty$.

Theorem 1.1. *If $0 < m < \infty$ and $\Omega \subset \mathbb{C}^m$ is open, then every $\mathcal{O}$-homomorphism $\mathcal{O}^E \to \mathcal{O}^F$ of plain sheaves is plain.*

This came as a surprise, because it fails in the simplest of all cases, when $m = 0$. This was pointed out by Lempert. When $\Omega = \mathbb{C}^0 = \{0\}$, $\mathcal{O}^E$, resp. $\mathcal{O}^F$, are identified with E and F, and the difference between $\mathcal{O}$-homomorphism and plain homomorphism boils down to the difference between linear and continuous linear operators $E \to F$. It would be interesting to decide whether Theorem 1.1 remains true if $\mathbb{C}^m$ is replaced by a Banach space.

Lempert observed that a variant of the original proof of Theorem 1.1 gives the corresponding theorem about local modules, and we shall derive Theorem 1.1 from it:

Theorem 1.2. *If $0 < m < \infty$ and $\zeta \in \mathbb{C}^m$, then every $\mathcal{O}_\zeta$-homomorphism of plain modules $\mathcal{O}^E_\zeta \to \mathcal{O}^F_\zeta$ is plain.*

Next we turn to the structure of $\mathcal{O}$-homomorphisms from plain sheaves $\mathcal{O}^E$ to a coherent sheaf $\mathcal{S}$. On the level of stalks, such homomorphisms have a simple description; however, this description applies only if the depth of the stalk $\mathcal{S}_\zeta$ is positive, a condition that corresponds to the positivity of m in Theorems 1.1 and 1.2. For our purposes we can define depth as follows. Let $\mathfrak{m}_\zeta \subset \mathcal{O}_\zeta$ denote the maximal ideal consisting of germs that vanish at ζ; assume $\mathfrak{m}_\zeta \neq 0$. For a finitely generated $\mathcal{O}_\zeta$-module M, $\operatorname{depth}_{\mathfrak{m}_\zeta} M = 0$ if M has a nonzero submodule N such that $\mathfrak{m}_\zeta N = 0$ (see Definition 4.1 and Proposition 4.2). Otherwise $\operatorname{depth}_{\mathfrak{m}_\zeta} M > 0$.

Theorem 1.3. *Let $\zeta \in \mathbb{C}^m$, M a finite $\mathcal{O}_\zeta$-module with $\operatorname{depth}_{\mathfrak{m}_\zeta} M > 0$, and $p : \mathcal{O}^n_\zeta \to M$ an epimorphism. Then, any $\mathcal{O}_\zeta$-homomorphism $\phi : \mathcal{O}^E_\zeta \to M$ factors through p, i.e., $\phi = p\psi$ with an $\mathcal{O}_\zeta$-homomorphism $\psi : \mathcal{O}^E_\zeta \to \mathcal{O}^n_\zeta$.*

Since our depth condition eliminates the possibility of a nonzero sheaf when $m = 0$, the above ψ is then induced by a germ in $\mathcal{O}^{\mathrm{Hom}(E,F)}_\zeta$. The depth condition is in fact necessary as shown in Theorem 5.1.

A global version of the theorem concerning epimorphisms $p : \mathcal{O}^n \to \mathcal{S}$ on a coherent sheaf, also holds, but we shall discuss it only in Section 8, since it depends on result of Lempert that has not yet been published. Theorem 1.3 can be recast in the language of analytic structure on sheaves, as defined in [LP07]; it says that analytic structures of coherent sheaves are unique. This will be explained in Section 7, along with the following corollary of Theorem 1.2:

Corollary 1.4. *If $\zeta \in \mathbb{C}^m$, $0 < m < \infty$, and E is an infinite dimensional Banach space, then the plain module $\mathcal{O}^E_\zeta$ is not free; it cannot even be embedded in a free module.*

2. Background

Here we quickly review a few notions of complex analysis. For more see [GR84, Muj86, Ser55]. Let X, E be Banach spaces (always over $\mathbb{C}$) and $\Omega \subset X$ open.

Definition 2.1. A function $f : \Omega \to E$ is holomorphic if for all $x \in \Omega$ and $\xi \in X$

$$df(x, \xi) = \lim_{\lambda \to 0} \frac{f(x + \xi\lambda) - f(x)}{\lambda}$$

exists, and depends continuously on $(x, \xi) \in \Omega \times X$.

If $X = \mathbb{C}^m$ with coordinates $(z_1, \dots, z_m)$, then this is equivalent to requiring that in some neighborhood of each $a \in \Omega$ one can expand f in a uniformly convergent power series

$$f = \sum_J e_J (z-a)^J, \qquad e_J \in E,$$

where multi-index notation is used. For general X one can only talk about homogeneous expansion. Recall that a function P between vector spaces V, W is an n-homogeneous polynomial if $P(v) = l(v, v, \dots, v)$ where $l : V^n \to W$ is an n-linear map. Given a ball $B \subset X$ centered at $a \in X$, any holomorphic $f : B \to E$ can be expanded in a series

$$f(x) = \sum_{n=0}^{\infty} P_n(x-a), \qquad x \in B, \tag{2.1}$$

where the $P_n : X \to E$ are continuous n-homogeneous polynomials. The homogeneous components P_n are uniquely determined, and the series (2.1) converges locally uniformly on B.

We denote by f_x the germ at $x \in \Omega$ of a function $\Omega \to E$, and by $\mathcal{O}^E$ the sheaf of germs of holomorphic functions $U \to E$, where $U \subset \Omega$ is open. The sheaf $\mathcal{O}^{\mathbb{C}} = \mathcal{O}$ is a sheaf of rings over Ω, and $\mathcal{O}^E$ is, in an obvious way, a sheaf of $\mathcal{O}$-modules. The sheaves $\mathcal{O}^E$ are called plain sheaves, and their stalks $\mathcal{O}^E_x$ plain modules. When $E = \mathbb{C}^n$, we write $\mathcal{O}^n$, resp. $\mathcal{O}^n_x$ for $\mathcal{O}^E$, resp. $\mathcal{O}^E_x$.

As said in the Introduction, $\mathrm{Hom}(E, F)$ denotes the space of continuous linear operators between Banach spaces E and F, endowed with the operator norm. Any holomorphic function $\Phi : \Omega \to \mathrm{Hom}(E, F)$ induces an $\mathcal{O}$-homomorphism $\mathcal{O}^E \to \mathcal{O}^F$ and any $\Psi \in \mathcal{O}_x^{\mathrm{Hom}(E,F)}$ induces an $\mathcal{O}_x$-homomorphism $\mathcal{O}^E_x \to \mathcal{O}^F_x$. The homomorphisms obtained in this manner are called plain homomorphisms.

3. Homomorphisms of Plain Sheaves and Modules

We shall deduce Theorem 1.2 from a weaker variant, which, however, is valid in an arbitrary Banach space:

Theorem 3.1. *Let X, E, F be Banach spaces,* $\dim X > 0$. *Let $\zeta \in X$ and $\phi : \mathcal{O}^E_\zeta \to \mathcal{O}^F_\zeta$ an $\mathcal{O}_\zeta$-homomorphism. Then there is a plain homomorphism $\psi : \mathcal{O}^E_\zeta \to \mathcal{O}^F_\zeta$ that agrees with ϕ on constant germs.*

We need two auxiliary results to prove this.

Proposition 3.2. *Let X, G be Banach spaces and $\pi_n : X \to G$ continuous homogeneous polynomials of degree $n = 0, 1, 2, \dots$. If for every $x \in X$ there is an $\epsilon_x > 0$ such that* $\sup_n \|\pi_n(\epsilon_x x)\| < \infty$, *then there is an $\epsilon > 0$ such that*

$$\sup_n \sup_{\|x\| < \epsilon} \|\pi_n(x)\| < \infty.$$

Here, and in the following, we indiscriminately use $\|\cdot\|$ for the norms on X, G, and whatever Banach spaces we encounter.

Proof. For numbers A and δ, consider the closed sets

$$X_{A,\delta} = \{x \in X : \sup_n \|\pi_n(\delta x)\| \le A\}.$$

By Baire's theorem $X_{A,\delta}$ contains a ball $\{x_0 + y : \|y\| < r\}$ for some $A, \delta, r > 0$. As a consequence of the polarization formula [Muj86, 1:10],

$$\pi_n(\xi) = \sum_{\sigma_j=\pm 1} \frac{\sigma_1 \ldots \sigma_n}{2^n n!} \pi_n(\delta x_0 + \sigma_1 \xi + \ldots + \sigma_n \xi), \qquad \text{for } \xi \in X,$$

see also [Muj86, Exercise 2M]. Therefore if $\|\xi\| < \delta r/n$, then $\pi_n(\xi) \leq A/n!$, and by homogeneity, for $\|x\| < \delta r/e$

$$\|\pi_n(x)\| = n^n e^{-n} \|\pi_n(ex/n)\| \leq A n^n/(e^n n!) \leq A.$$

□

Proposition 3.3. *Let X, E, F be Banach spaces, $\Omega \subset X$ open, and $g : \Omega \to \mathrm{Hom}(E, F)$ a function. If for every $v \in E$ the function $gv : X \to F$ is holomorphic, then g itself is holomorphic.*

Proof. This is Exercise 8.E in [Muj86]. First one shows using the Principle of Uniform Boundedness that g is locally bounded. Standard one variable Cauchy representation formulas then show g is continuous and ultimately holomorphic. □

Proof of Theorem 3.1. If $v \in E$ we write $\tilde{v} \in \mathcal{O}^E_\zeta$ for the constant germ whose value is v. Without loss of generality we can take $\zeta = 0$. Let the germ $\phi(\tilde{v}) \in \mathcal{O}^F_0$ have homogeneous series

$$\sum_{n=0}^{\infty} P_n(x, v). \tag{3.1}$$

Thus, P_n is $\mathbb{C}$-linear in v, and for fixed v, $P_n(\cdot, v)$ is a continuous n-homogeneous polynomial. For each $v \in E$ (3.1) converges if $\|x\|$ is sufficiently small.

Now let $\lambda \in \mathrm{Hom}(X, \mathbb{C})$, and suppose that with $v_j \in E$ the series $\sum_{i=0}^{\infty} v_i \lambda^i$ represents a germ $e \in \mathcal{O}^E_0$. For example, this will be the case if the v_i are unit vectors. With an arbitrary $N \in \mathbb{N}$ and some $f \in \mathcal{O}^F_0$

$$\begin{aligned} \phi(e) &= \sum_{i<N} \phi(\tilde{v})\lambda^i + \lambda^N f \\ &= \sum_{j<N} \sum_{n=0}^{\infty} P_n(\cdot, v_i)\lambda^i + \lambda^N f \\ &= \sum_{j<N} \sum_{n=0}^{j} P_n(\cdot, v_{j-n})\lambda^{j-n} + \lambda^N g, \qquad \text{where } g \in \mathcal{O}^F_0. \end{aligned}$$

Hence the homogeneous components of $\phi(e)$ are

$$Q_j(x) = \sum_{n=0}^{j} P_n(x, v_{j-n})\lambda^{j-n}(x), \qquad j = 0, 1, 2, \ldots \tag{3.2}$$

We use this to prove, by induction on n, that for any $x \in X$ the map $v \mapsto P_n(x, v)$ is not only linear but also continuous.

Suppose this is true for $n < k$. Take an $x \in X$, which can be supposed to be nonzero, and $\lambda \in \mathrm{Hom}(X, \mathbb{C})$ so that $\lambda(x) = 1$. If $v \mapsto P_k(x, v)$ were not continuous,

we could inductively select unit vectors $v_i \in E$ so that

$$\|P_k(x, v_{j-k})\| > \sum_{n=0}^{k-1} \|P_n(x, \cdot)\| + j^j + \sum_{n=k+1}^{j} \|P_n(x, v_{j-n})\|,$$

for $j = k, k+1, \dots$. Here $\|P_n(x, \cdot)\|$ stands for the operator norm of the homomorphism $P_n(x, \cdot) \in \mathrm{Hom}(E, F)$, $n < k$. However, (3.2) would then imply

$$\|Q_j(x)\| \geq \|P_k(x, v_{j-k}))\| - \sum_{n=0}^{k-1} \|P_n(x, \cdot)\| - \sum_{n=k+1}^{j} \|P_n(x, v_{j-n})\| > j^j,$$

which would preclude $\sum Q_j$ from converging in any neighborhood of $0 \in X$. The contradiction shows that $P_k(x, \cdot) \in \mathrm{Hom}(E, F)$, in fact for every k and $x \in X$. Let us write $\pi_k(x)$ for $P_k(x, \cdot)$.

Now, for fixed $v \in E$, $P_n(\cdot, v) = \pi_n v$ is holomorphic. We can apply Proposition 3.3 to conclude that $\pi_n : X \to \mathrm{Hom}(E, F)$ is a holomorphic, n-homogeneous polynomial.

Next we estimate $\|\pi_n(x)\|$ for fixed $x \in X$. Suppose a sequence $\delta_n \geq 0$ goes to 0 super-exponentially, in the sense that $\delta_n = o(\epsilon^n)$ for all $\epsilon > 0$. Then for any homogeneous series $\sum p_n$ representing a germ $f \in \mathcal{O}_0^F$ we have $\sup_n \delta_n \|p_n(x)\| < \infty$.

In particular, $\sup_n \delta_n \|\pi_n(x) v\| < \infty$ for all $v \in E$, and by the Principle of Uniform Boundedness, $\delta_n \|\pi_n(x)\|$ is bounded. This being so, there is an $\epsilon = \epsilon_x > 0$ such that $\epsilon^n \|\pi_n(x)\|$ is bounded. Indeed, otherwise we could find $n_1 < n_2 < \dots$ so that

$$\|\pi_{n_t}(x)\| > t^{n_t}, \ t = 1, 2, \dots.$$

But then the sequence

$$\delta_n = \begin{cases} t^{-n_t/2} & \text{if } n = n_t, \\ 0, & \text{otherwise,} \end{cases}$$

would go to 0 super-exponentially and yet $\delta_{n_t} \|\pi_{n_t}(x)\| \to \infty$; a contradiction.

Thus, for each x we have found $\epsilon_x > 0$ so that $\sup_n \|\pi_n(\epsilon_x x)\|$ is bounded. By Proposition 3.2, the π_n are uniformly bounded on some ball $\{x : \|x\| < \epsilon\}$. Therefore the series

$$\sum_{n=0}^{\infty} \pi_n(x) = \Phi(x)$$

converges uniformly on some neighborhood of $0 \in X$, and represents a $\mathrm{Hom}(E, F)$-valued holomorphic function there. By the construction of $\pi_n(x)v = P_n(x, v)$, see (3.1), the plain homomorphism $\mathcal{O}_0^E \to \mathcal{O}_0^F$ induced by Φ agrees with ϕ on constant germs $\tilde{v}$, and the proof is complete. □

Proof of Theorem 1.2. In view of Theorem 3.1, all we have to show is that (for $X = \mathbb{C}^m, 0 < m < \infty$) if an $\mathcal{O}_\zeta$-homomorphism $\phi : \mathcal{O}_\zeta^E \to \mathcal{O}_\zeta^F$ annihilates constant germs then it is in fact 0. This we formulate in a slightly greater generality:

Lemma 3.4. *Let $\zeta \in \mathbb{C}^m$ and M be an $\mathcal{O}_\zeta$-module. If a homomorphism $\theta : \mathcal{O}_\zeta^F \to M$ annihilates all constant germs, then*

$$\mathrm{Im}\, \theta \subset \bigcap_{k=0}^{\infty} \mathfrak{m}_\zeta^k M. \tag{3.3}$$

Proof. Along with constants, θ will annihilate the $\mathcal{O}_\zeta$-module generated by constants, in particular, the polynomial germs. Since any $e \in \mathcal{O}_\zeta^E$ is congruent, modulo an arbitrary power of $\mathfrak{m}_\zeta^k$ of the maximal ideal to a polynomial, and furthermore, $\theta(\mathfrak{m}_\zeta^k \mathcal{O}_\zeta^E) \subset \mathfrak{m}_\zeta^k M$, (3.3) follows. $\square$

This then completes the proof of Theorem 1.2 since $\bigcap_{k=0}^\infty \mathfrak{m}_\zeta^k \mathcal{F}_\zeta = 0$. $\square$

Proof of Theorem 1.1. Let $\phi : \mathcal{O}^E \to \mathcal{O}^F$ be an $\mathcal{O}$-homomorphism. For $v \in E$ let $\hat{v} : \Omega \to \mathcal{O}^E$ be the section that associates with $z \in \Omega$ the germ at z of the constant function $e \equiv v$. Then $\phi(\hat{v})$ is a section of $\mathcal{O}^F$ and so there is a holomorphic function $f(\cdot, v) : \Omega \to F$ whose germs $f(\cdot, v)_z$ at various $z \in \Omega$ agree with $\phi(\tilde{v})(z)$. By Theorem 1.2 for each $\zeta \in \Omega$ we can find a germ $\Phi^\zeta \in \mathcal{O}_\zeta^{\mathrm{Hom}(E,F)}$ such that

$$f(\cdot, v)_\zeta = \Phi^\zeta v$$

Therefore for fixed ζ, $f(\zeta, \cdot) \in \mathrm{Hom}(E, F)$. Let $\Phi(\zeta) = f(\zeta, \cdot)$. Proposition 3.3 implies that $\Phi : \Omega \to \mathrm{Hom}(E, F)$ is holomorphic and by construction induces ϕ on constant germs.

This means that Φ^ζ above and the germ Φ_ζ of Φ induce homomorphisms $\mathcal{O}_\zeta^E \to \mathcal{O}_\zeta^F$ that agree on constant germs; since we are talking about plain homomorphisms, the two induced homomorphisms in fact agree. Hence ϕ is induced by Φ. $\square$

4. Depth Lemmas

The usual definition of depth, see [Eis99, pp. 423,429] or [Mat80, p. 130], gives the following

Definition 4.1. Let R be a Noetherian ring (always commutative, unital), $I \subset R$ and ideal and M a finite R-module, such that $M \neq IM$. We say the I-depth of M, $\operatorname{depth}_I M$, is positive if there is a nonzerodivisor r on M with $r \in I$ and $rM \neq M$; otherwise the I-depth is 0.

If $M = IM$, in particular, if $M = 0$, the convention is that the I-depth is positive infinity.

We note that, when R is a field, the maximal ideal is $\mathfrak{m} = 0$ and so, if M is a finite R-module, then the $\mathfrak{m}$-depth of M is positive (infinity) if and only if $M = 0$. When R is not a field, there is an alternative criterion for the positivity of the depth. While this lemma is not new, we include it for the sake completeness:

Proposition 4.2. *For a Noetherian local ring $(R, \mathfrak{m})$, not a field, a finite R-module M has $\operatorname{depth}_\mathfrak{m} M = 0$ if and only if there is a nonzero submodule $L \subset M$ such that $\mathfrak{m}L = 0$.*

Proof. Assume $\operatorname{depth}_\mathfrak{m} M = 0$. Then, $M \neq \mathfrak{m}M$ and every $r \in \mathfrak{m} \backslash \{0\}$ is a zerodivisor on M. At this point we recall the notion of an associated prime: if R is a commutative ring and M is an R-module, then a prime ideal p of R is associated to M, if there is an $x \in M$ such that $p = \operatorname{ann} x$. We shall make use of the following fact: if R is a Noetherian ring and M is a finite R-module, then there are finitely many primes associated to M, and furthermore, each zerodivisor on M is contained in one of them, see [Eis99, Theorem 3.1].

Thus, in our setting, $\mathfrak{m}$ is a subset of the finite union of the associated primes. Now, the prime avoidance lemma [Eis99, Theorem 3.3] states that $\mathfrak{m}$ is contained

in one of the associated primes, and hence, itself is an associated prime. Therefore, $\mathfrak{m}x = 0$ for some nonzero $x \in M$ and we put $L = xR$.

Conversely, suppose $L \subset M$ is a nonzero submodule such that $\mathfrak{m}L = 0$. Then, every $r \in \mathfrak{m}\backslash\{0\}$ is a zerodivisor. Since $M \neq 0$ is a finite R-module, Nakayama's lemma [Eis99, Corollary 4.8] implies that $M \neq \mathfrak{m}M$. So, $\operatorname{depth}_{\mathfrak{m}} M = 0$. □

For the proof of Theorem 1.3 we shall need a number of Lemmas that are algebraic in nature. Recall the notion of localization at a prime. Suppose R is a ring, $p \subset R$ a prime ideal, and M an R-module. Consider the multiplicatively closed set $S = R\backslash p$, then the localization of M at p is

$$M_p = M \times S/\sim,$$

where $(v, s) \sim (w, t)$ means $q(vt - ws) = 0$ for some $q \in S$. Elements of M_p are written as fractions v/s. The usual rules for operating with fractions turn R_p into a ring and M_p into a module over it. Localization is a functor, in particular, a homomorphism $\alpha : M \to M'$ of R-modules induces a homomorphism $\alpha_p : M_p \to M'_p$.

Lemma 4.3. *Let R be a unique factorization domain and $(p) \subset R$ a principal prime ideal. If $N \subset R^m$ is a finite module, then there is a free submodule $F \subset N$ such that $F_{(p)} = N_{(p)}$.*

Proof. Any element of $R_{(p)}$ is either invertible or divisible by p. Since R is a UFD, it follows that given $u_1, \dots, u_k \in R^m_{(p)}$ any non-trivial linear relation

$$\sum r_j u_j = 0, \text{ with } r_j \in R_{(p)},$$

can be solved for some u_j. Hence finitely generated submodules of $R^m_{(p)}$ are free. In particular, $N_{(p)}$ has a free generating set (v_j/s_j), $j = 1, \dots, k$. We can therefore, take F to be the module generated by v_j's. □

Lemma 4.4. *Let $(R, \mathfrak{m})$ be a local ring which is a unique factorization domain, and Q its field of fractions. Let $\rho : A \to B$ be a homomorphism of finite free R-modules. If $\operatorname{depth}_{\mathfrak{m}} \operatorname{coker} \rho > 0$, then there are a finite free R-module C and a homomorphism $\theta : B \to C$ such that*

(i) $\ker \rho = \ker \theta\rho$,
(ii) $\operatorname{coker} \theta\rho$ *is zero or has positive* $\mathfrak{m}$*-depth,*
(iii) $(\operatorname{coker} \theta\rho) \otimes Q = 0$.

Proof. It will be convenient to assume, as we may, that $A = \operatorname{Hom}(R^m, R)$, similarly $B = \operatorname{Hom}(R^n, R)$, and ρ is the transpose of a homomorphism $\pi : R^n \to R^m$. Set $N = \operatorname{Im} \pi$. With a principal prime ideal $(p) \subset R$ to be chosen later let $F \subset N$ be as in Lemma 4.3. Define a homomorphism $\sigma : F \to R^n$, by specifying its values on a free generating set, so that $\pi \circ \sigma$ is the inclusion $\iota : F \hookrightarrow R^m$. We will show that with a suitable choice of p we can take

$$\theta = \sigma^* : \operatorname{Hom}(R^n, R) = B \to \operatorname{Hom}(F, R) = C.$$

We summarize the homomorphisms in question in the following commutative diagram

$$\begin{tikzcd} F \arrow[r, hook, "\iota"] \arrow[d, "\sigma"'] \arrow[dr, hook] & R^m \\ R^n \arrow[r, two heads, "\pi"] & N \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} C & A \arrow[l, "\iota^*"'] \arrow[d] \\ B \arrow[u, "\theta"] & \mathrm{Hom}(N,R). \arrow[l, hook', "\rho"'] \arrow[ul] \end{tikzcd}$$

First note that, localizing

$$\begin{aligned} &\operatorname{Im}\pi_{(p)} = N_{(p)} = F_{(p)} = \operatorname{Im}\iota_{(p)}, \text{ whence} \\ &\ker\rho_{(p)} = \ker\pi^*_{(p)} = \ker\iota^*_{(p)} = \ker\theta_{(p)}\rho_{(p)} \subset A_{(p)}. \end{aligned} \tag{4.1}$$

Pulling back by the injective localization map $A \to A_p$ we obtain (i).

Next, $N \otimes Q$ is the image of the vector space homomorphism $\pi \otimes \mathrm{id}_Q$, while $(\operatorname{Im}\rho) \otimes Q$ is the image of its transpose $\rho \otimes \mathrm{id}_Q$. It follows that they have the same dimension. Since $N \otimes Q = N_{(p)} \otimes Q = F_{(p)} \otimes Q = F \otimes Q$ by Lemma 4.3, $(\operatorname{Im}\rho) \otimes Q$, $F \otimes Q$, and so $C \otimes Q$ have the same dimensions. As $\theta \otimes \mathrm{id}_Q$ restricts to a homomorphism $(\operatorname{Im}\rho) \otimes Q \to C \otimes Q$ of equidimensional vector spaces, and by (i) it is injective, (iii) follows.

To achieve (ii), we note that if R is a field, then it suffices to set $p = 0$, for, in that case, $\operatorname{coker}\theta\rho = (\operatorname{coker}\theta\rho) \otimes Q = 0$. So, we will assume that R is not a field. We pick a nonzerodivisor $r \in R$ on $(\operatorname{coker}\rho)$, (see Definition 4.1), and let p be one of its prime divisors. Thus, $p \in \mathfrak{m}$ is a nonzerodivisor on $\operatorname{coker}\rho$. We claim it is a nonzerodivisor on $\operatorname{coker}\theta\rho$ as well.

Suppose p multiplies the class in $\operatorname{coker}\theta\rho$ of a $\gamma \in C$ into 0. This means that $p\gamma = \theta\rho\alpha = \iota^*\alpha$ with some $\alpha \in A = \mathrm{Hom}(R^m, R)$. It follows that the values that $\iota^*\alpha$, resp. $\iota^*_{(p)}\alpha_{(p)}$, take are divisible by p in R, resp. $R_{(p)}$. By (4.1) $\pi^*_{(p)}\alpha_{(p)}$ takes the same values as $\iota^*_{(p)}\alpha_{(p)}$. Now for any $s \in R$, p divides s in R precisely when p divides $(s/1)$ in $R_{(p)}$; therefore $\pi^*\alpha = \rho\alpha$ is divisible by p, say

$$\rho\alpha = p\beta, \ \beta \in B. \tag{4.2}$$

Thus, $p\gamma = \theta\rho\alpha = p\,\theta\beta$ and $\gamma = \theta\beta$. On the other hand, (4.2) shows that p multiplies the class of β in $\operatorname{coker}\rho$ into 0. By our choice of p, this implies the class of β is already 0, i.e., $\beta = \operatorname{Im}\rho$. Hence $\gamma = \theta\beta \in \operatorname{Im}\theta\rho$, and the class of γ in $\operatorname{coker}\theta\rho$ is 0. Thus, p is indeed a nonzerodivisor on $\operatorname{coker}\theta\rho$.

But then we are done, since, by Nakayama's lemma $\mathfrak{m}\operatorname{coker}\theta\rho \neq \operatorname{coker}\theta\rho$, unless $\operatorname{coker}\theta\rho = 0$. Therefore, $\operatorname{coker}\theta\rho$ has positive $\mathfrak{m}$-depth. $\square$

In the next lemma we use the following notation. As before, $\mathcal{O}_0$ is the local ring at $0 \in \mathbb{C}^m$, $m \geq 1$. The subring of germs independent of the last coordinate z_m of $z \in \mathbb{C}^m$ is denoted $\mathcal{O}'_0$; the maximal ideals in $\mathcal{O}_0$, $\mathcal{O}'_0$ are $\mathfrak{m}$, $\mathfrak{m}'$. Any $\mathcal{O}_0$-module is automatically an $\mathcal{O}'_0$-module.

Lemma 4.5. *Suppose $h \in \mathcal{O}_0$ is the germ of a Weierstrass polynomial and M a finite $\mathcal{O}_0$-module such that $hM = 0$. Then* $\operatorname{depth}_{\mathfrak{m}} M = 0$ *if and only if* $\operatorname{depth}_{\mathfrak{m}'} M = 0$.

Proof. We note first that M is a finite $\mathcal{O}'_0$-module. Indeed, it is an $\mathcal{O}_0/h\mathcal{O}_0$-module, and finite as such. Since $\mathcal{O}_0/h\mathcal{O}_0$ is also finitely generated as an $\mathcal{O}'_0$-module, our claim follows.

If $M = 0$, then $\operatorname{depth}_{\mathfrak{m}} M = \operatorname{depth}_{\mathfrak{m}'} M = \infty$. So, we will assume $M \neq 0$. Suppose first that $\operatorname{depth}_{\mathfrak{m}'} M = 0$. We claim that there is a nonzero $u \in M$ such that $\mathfrak{m}' u = 0$. Indeed, this is obvious when $m = 1$, since $\mathcal{O}'_0$ is a field. On the other hand, when $m \geq 2$, we arrive at this conclusion by applying Proposition 4.2.

Write $h = z_m^d + \sum_{j=0}^{d-1} a_j z_m^j$, where $a_j \in \mathfrak{m}'$, $d > 0$. As $u \neq 0$ but

$$z_m^d u = hu - \sum_{j=0}^{d-1} a_j u z_m^j = 0,$$

there is a largest $k = 0, 1, \dots, k-1$ such that $v = z_m^k u \neq 0$. Then $z_m v = 0$, whence $\mathfrak{m} v = 0$, and we conclude by Proposition 4.2, (note, $\mathcal{O}_0$ is not a field).

Conversely, suppose that $\operatorname{depth}_{\mathfrak{m}} M = 0$. By Proposition 4.2, there is a nonzero submodule $L \subset M$ with $\mathfrak{m} L = 0$. We claim that $\operatorname{depth}_{\mathfrak{m}'} M = 0$. Indeed, since $\mathfrak{m}' \subset \mathfrak{m}$, this is a consequence of Proposition 4.2 when $m \geq 2$. On the other hand, this is obvious if $m = 1$, for in this case, $\mathcal{O}'_0$ is a field and $M \neq 0$. $\square$

5. The Proof of Theorem 1.3

Let us write (T_m) for the statement of Theorem 1.3, to indicate the number of variables involved. We prove it by induction on $m \geq 0$. When $m = 0$, the depth assumption does not hold, unless $M = 0$, and so, the claim is obvious.

When $m = 1$, $\mathcal{O}_\zeta$ is a principal ideal domain, and by the corresponding structure theorem M is a direct sum of a free module and modules of form $\mathcal{O}_\zeta/\mathfrak{m}_\zeta^k$, see [DF99, Chapter 12, Theorem 5]. If M contained a submodule of the latter type, then by Proposition 4.2 it would have $\mathfrak{m}$-depth 0 (take $L = \mathfrak{m}_\zeta^{k-1}/\mathfrak{m}_\zeta^k \subset \mathcal{O}_\zeta/\mathfrak{m}_\zeta^k$). Therefore M is free, p has a right inverse q and $\psi = q\phi$ will do.

Now assume (T_{m-1}) holds for some $m \geq 2$, and prove (T_m). We are free to take $\zeta = 0$. Let Q be the field of fractions of $\mathcal{O}_0$. We first verify (T_m) for torsion modules M, i.e., those for which $M \otimes Q = 0$.

Since each generator $v \in M$ is annihilated by some nonzero $h_v \in \mathcal{O}_0$, there is a nonzero $h \in \mathcal{O}_0$ that annihilates all of M. We can assume h is (the germ of) a Weierstrass polynomial of degree $d \geq 1$ in z_m. We write $z = (z', z_m)$ for $z \in \mathbb{C}^m$, and let $\mathcal{O}'_0, \mathcal{O}'^F_0$ denote the ring/module of the corresponding germs in $\mathbb{C}^{m-1}$. As before we embed $\mathcal{O}'_0 \subset \mathcal{O}_0$, $\mathcal{O}'^F_0 \subset \mathcal{O}^F_0$. This makes any $\mathcal{O}_0$-module an $\mathcal{O}'_0$-module, and any homomorphism $\phi : N_1 \to N_2$ of $\mathcal{O}_0$-modules descends to an $\mathcal{O}'_0$-homomorphism

$$\phi' : N_1/hN_1 \to N_2/hN_2. \tag{5.1}$$

As an example, a version of Weierstrass' division theorem remains true for holomorphic germs valued in a Banach space F (the proof in [GR84] applies). Concretely, we can write any $f \in \mathcal{O}^F_0$ uniquely as

$$f = hf_0 + \sum_{j=0}^{d-1} f'_j z_m^j, \qquad f_0 \in \mathcal{O}^F_0, f'_j \in \mathcal{O}'^F_0. \tag{5.2}$$

Clearly, the $\mathcal{O}'_0$-homomorphism

$$\mathcal{O}^F_0 \ni f \mapsto (f'_0, \dots, f'_{d-1}) \in (\mathcal{O}'^F_0)^{\oplus d} \tag{5.3}$$

descends to an isomorphism

$$\mathcal{O}^F_0/h\mathcal{O}^F_0 \xrightarrow{\approx} (\mathcal{O}'^F_0)^{\oplus d} \tag{5.4}$$

of $\mathcal{O}'_0$-modules. Composing this with the embedding

$$(\mathcal{O}'^F_0)^{\oplus d} \ni (f'_j) \mapsto \sum_j f'_j z^j_m \in \mathcal{O}^F_0, \tag{5.5}$$

we obtain an $\mathcal{O}'_0$-homomorphism

$$\mathcal{O}^F_0/h\mathcal{O}^F_0 \to \mathcal{O}^F_0, \tag{5.6}$$

which is a right inverse of the canonical projection $\mathcal{O}^F_0 \to \mathcal{O}^F_0/h\mathcal{O}^F_0$.

Now $p : \mathcal{O}^n_0 \to M$ and $\phi : \mathcal{O}^E_0 \to M$ of the theorem induce $\mathcal{O}'_0$-homomorphisms

$$p' : \mathcal{O}^n_0/h\mathcal{O}^n_0 \to M,\ \phi' : \mathcal{O}^E_0/h\mathcal{O}^E_0 \to M,$$

as in (5.1), remembering that $hM = 0$. Clearly, p' is surjective. Also, by Lemma 4.5 $\mathrm{depth}_{\mathfrak{m}'} M > 0$. Because of the isomorphism (5.4), (T_{m-1}) implies there is an $\mathcal{O}'_0$-homomorphism

$$\bar{\chi} : \mathcal{O}^E_0/h\mathcal{O}^E_0 \to \mathcal{O}^n_0/h\mathcal{O}^n_0$$

such that $\phi' = p'\bar{\chi}$. Since the projection $\mathcal{O}^n_0 \to \mathcal{O}^n_0/h\mathcal{O}^n_0$ has a right inverse, cf. (5.6), $\bar{\chi}$ is induced by an $\mathcal{O}'_0$-homomorphism $\chi : \mathcal{O}^E_0 \to \mathcal{O}^n_0$, which then satisfies $\phi = p\,\chi$. All that remains is to replace χ by an $\mathcal{O}_0$-homomorphism ψ, which we achieve as follows.

If a holomorphic germ, say f, at 0 valued in a Banach space $(F, \|\cdot\|_F)$ has a representative on a connected neighborhood V of 0, we write

$$[f]_V = \sup_{v\in V} \|f(v)\|_F \le \infty,$$

where f on the right stands for the representative. Now consider the composition of $\chi|_{\mathcal{O}'^E_0}$ with (5.3), where $F = \mathbb{C}^n$,

$$\mathcal{O}'^E_0 \xrightarrow{\chi} \mathcal{O}^n_0 \to \mathcal{O}'^{nd}_0. \tag{5.7}$$

By Theorem 1.1 this $\mathcal{O}'_0$-homomorphism is induced by a $\mathrm{Hom}(E, \mathbb{C}^{nd})$-valued holomorphic function, defined on some neighborhood U of $0 \in \mathbb{C}^{m-1}$. It follows that if $V' \Subset U$ and $V'' \Subset \mathbb{C}$ are connected neighborhoods of $0 \in \mathbb{C}^{m-1}$, resp. $0 \in \mathbb{C}$, then there is a constant C such that for each $e' \in \mathcal{O}'^E_0$ that has a representative defined on V'

$$[\chi(e')]_{V'\times V''} \le C[e']_{V'}. \tag{5.8}$$

Indeed, χ is obtained by composing (5.7) with (5.5) (again, $F = \mathbb{C}^n$), and this latter is trivial to estimate.

Now define $\psi : \mathcal{O}^E_0 \to \mathcal{O}^n_0$ by $\psi(e) = \sum_{j=0}^\infty \chi(e'_j)z^j_m$, if $e = \sum_{k=0}^\infty e'_j z^j_m \in \mathcal{O}^E_0$. Cauchy estimates for e'_j and (5.8) together imply that the series above indeed represents a germ $\psi(e) \in \mathcal{O}^n_0$. It is straightforward that ϕ is an $\mathcal{O}_0$-homomorphism. Because of this, $p\psi = \phi$ holds on $h\mathcal{O}^E_0$, both sides being zero. It also holds on polynomials $e = \sum_{j=0}^k e'_j z^j_m$, as

$$(p\psi)(e) = p\sum_j \chi(e'_j)z^j_m = \sum_j \phi(e'_j)z^j_m = \phi(e).$$

The division formula (5.2), this time with $F = E$, now implies $p\psi = \phi$ on all $\mathcal{O}^E_0$.

Having taken care of torsion modules, consider a general module M as in the theorem. Since $\mathcal{O}_0$ is Noetherian, $\ker p$ is finitely generated; let

$$\rho : \mathcal{O}^r_0 \to \mathcal{O}^n_0$$

have image $\ker p$. So, $M \approx \operatorname{coker} \rho$. Construct a free $\mathcal{O}_0$-module C together with a homomorphism $\theta : \mathcal{O}_0^n \to C$ as in Lemma 4.4. Let $\pi : C \to \operatorname{coker} \theta\rho$ be the canonical projection. Here is a diagram to keep track of all the homomorphisms in question:

$$\begin{tikzcd}
\mathcal{O}_0^r \arrow[d, "\rho"'] & & \\
\mathcal{O}_0^n \arrow[r, "\theta"] \arrow[d, "p"'] & C \arrow[r, "\pi"] & \operatorname{coker} \theta\rho \\
M & & \mathcal{O}_0^E . \arrow[ll, "\phi"'] \arrow[ul, dashed, "\tilde{\psi}"] \arrow[u, "\tilde{\phi}"']
\end{tikzcd}$$

We are yet to introduce $\tilde{\phi}$, $\tilde{\psi}$. For $e \in \mathcal{O}_0^E$ choose $v \in \mathcal{O}_0^n$ so that $p(v) = \phi(e)$. Then $\pi\theta(v)$ is independent of which v we choose, since any two choices differ by an element of $\ker p = \operatorname{Im} \rho$, which $\pi\theta$ then maps to 0. We let $\tilde{\phi}(e) = \pi\theta(v)$. We want to lift $\tilde{\phi}$ to C; this certainly can be done if $\operatorname{coker} \theta\rho = 0$. Otherwise Lemma 4.4 guarantees that $\operatorname{depth}_{\mathfrak{m}} \operatorname{coker} \theta\rho > 0$ and $(\operatorname{coker} \theta\rho) \otimes Q = 0$. Hence we can apply the first part of this proof to obtain a homomorphism $\tilde{\psi} : \mathcal{O}_0^E \to C$ such that $\pi\tilde{\psi} = \tilde{\phi}$.

Finally, we lift $\tilde{\psi}$ to $\mathcal{O}_0^n$ as follows. For $e \in \mathcal{O}_0^E$ choose $v \in \mathcal{O}_0^n$ and $w \in \mathcal{O}_0^r$ so that $\phi(e) = p(v)$ and $\tilde{\psi}(e) = \theta(v) + \theta\rho(w)$. Again $v + \rho(w) \in \mathcal{O}_\zeta^n$ is independent of the choices. (It suffices to verify this for $e = 0$. Then $v \in \ker p = \operatorname{Im} \rho$; let $v \in \rho(u)$, $u \in \mathcal{O}_0^r$. Hence $0 = \theta(v) + \theta\rho(w) = \theta\rho(u + w)$. By Lemma 4.4 this implies $0 = \rho(u + w) = v + \rho w$ as claimed.)

Therefore we can define a homomorphism $\psi : \mathcal{O}_0^E \to \mathcal{O}_0^n$ by letting $\psi(e) = v + \rho w$. Since $p\psi(e) = p(v) = \phi(e)$, ψ is the homomorphism we were looking for.

We conclude this section by showing that the depth condition in Theorem 1.3 is also necessary.

Theorem 5.1. *Let $m, n \geq 1$, $\zeta \in \mathbb{C}^m$, $M \neq 0$ a finite $\mathcal{O}_\zeta$-module, and $p : \mathcal{O}_\zeta^n \to M$ an epimorphism. If $\operatorname{depth}_{\mathfrak{m}_\zeta} M = 0$, then for any infinite dimensional Banach space E there is a homomorphism $\phi : \mathcal{O}_\zeta^E \to M$ that does not factor through p.*

Proof. By Proposition 4.2, M has a submodule $N \neq 0$ for which $\mathfrak{m}_\zeta N = 0$. This implies that N is a finite dimensional vector space over $\mathbb{C} \subset \mathcal{O}_\zeta$ ($\mathbb{C}$ embedded as constant germs). Consider the homomorphism of $\mathbb{C}$-vector spaces

$$\epsilon : \mathcal{O}_\zeta^E \to E, \ \epsilon(e) = e(0),$$

and a $\mathbb{C}$-linear map $l : E \to N$. The composition $l\epsilon : \mathcal{O}_\zeta^E \to N$ is a homomorphism of $\mathcal{O}_\zeta$-modules. If it can be written as $p\psi$, where $\psi : \mathcal{O}_\zeta^E \to \mathcal{O}_\zeta^n$ is a homomorphism of$\mathcal{O}_\zeta$-modules, then ψ is plain by Theorem 1.2 and, by looking at how ψ acts on constant germs, we see that l must be continuous. Therefore, by taking $l : E \to N$ to be linear and discontinuous, we obtain a nonfactorizable homomorphism $\psi = l\epsilon$. $\square$

6. A Local Theorem

Theorem 6.1. *Let $\mathcal{S}$ be a coherent sheaf over an open $\Omega \subset \mathbb{C}^m$, such that the $\mathfrak{m}_\zeta$-depth of each nonzero stalk is positive. Suppose $p : \mathcal{O}^n \to \mathcal{S}$ is an epimorphism and E is a Banach space. Then any $\mathcal{O}$-homomorphism $\mathcal{O}^E \to \mathcal{S}$ factors through p in some neighborhood of ζ.*

This theorem is a special case of a stronger, global, statement, whose proof, however, involves a cohomology vanishing theorem for infinitely generated sheaves. Since the local version yields a few immediate applications, we will post-pone the proof of the stronger theorem until Section 8.

The local statement is a simple consequence of Theorem 1.2 once the following two auxiliary statements are proved:

Lemma 6.2. *Let $\rho : \mathcal{A} \to \mathcal{B}$ be a homomorphism of finite free $\mathcal{O}$-modules over $\Omega \subset \mathbb{C}^m$ and $\zeta \in \Omega$. Denote by Q the field of fractions of $\mathcal{O}_\zeta$. Then, there are a finite free $\mathcal{O}$-module $\mathcal{C}$, a neighborhood U of ζ, and a homomorphism $\theta : \mathcal{B}|_U \to \mathcal{C}|_U$, such that*

(i) $\ker \rho|_U = \ker(\theta\rho)|_U$,
(ii) $(\operatorname{coker} \theta\rho)_\zeta \otimes Q = 0$.

Proof. We just repeat the proof of Lemma 4.4. Let $\mathcal{N} = \operatorname{Im} \rho^*$, where

$$\rho^* : \mathbf{Hom}(\mathcal{B}, \mathcal{O}) \to \mathbf{Hom}(\mathcal{A}, \mathcal{O})$$

is the map dual to ρ. Since $\mathcal{N}_\zeta \otimes Q$ is a finite dimensional vector space over Q, there is a finite free $\mathcal{O}_\zeta$-module $\mathcal{F}_\zeta \hookrightarrow \mathcal{N}_\zeta$ such that $\mathcal{F}_\zeta \otimes Q = N_\zeta \otimes Q$. Define $\sigma_\zeta : \mathcal{F}_\zeta \to \operatorname{Hom}(\mathcal{B}_\zeta, \mathcal{O}_\zeta)$ by specifying its values on a free generator set so that $\rho^*_\zeta \sigma_\zeta$ is the inclusion $\iota_\zeta : \mathcal{F}_\zeta \hookrightarrow \operatorname{Hom}(\mathcal{A}_\zeta, \mathcal{O}_\zeta)$. Take

$$\theta_\zeta = \sigma^*_\zeta : \mathcal{B}_\zeta \approx \operatorname{Hom}(\operatorname{Hom}(\mathcal{B}_\zeta, \mathcal{O}_\zeta), \mathcal{O}_\zeta) \to \operatorname{Hom}(\mathcal{F}_\zeta, \mathcal{O}_\zeta) = \mathcal{C}_\zeta.$$

Now, since $\mathcal{C}_\zeta$ is a free $\mathcal{O}_\zeta$-module, we can consider it as a stalk of a sheaf $\mathcal{C}$ of finite free $\mathcal{O}$-modules. The stalk homomorphism $\theta_\zeta : \mathcal{B}_\zeta \to \mathcal{C}_\zeta$ is induced by a homomorphism-valued holomorphic map $\bar{\theta}$ on some neighborhood U of ζ. Hence θ_ζ extends to an $\mathcal{O}$-homomorphism $\theta : \mathcal{B}|_U \to \mathcal{C}|_U$.

Since $\otimes$ is a right-exact covariant functor,

$$\operatorname{Im}(\rho^*_\zeta \otimes \mathrm{id}_Q) = \mathcal{N}_\zeta \otimes Q = \mathcal{F}_\zeta \otimes Q = \operatorname{Im}(\iota_\zeta \otimes \mathrm{id}_Q),$$

and so,

$$\ker(\rho_\zeta \otimes \mathrm{id}_Q) = \ker(\iota^*_\zeta \otimes \mathrm{id}_Q) = \ker(\theta\rho)_\zeta \otimes \mathrm{id}_Q.$$

Noting that $\mathcal{A}_\zeta \to \mathcal{A}_\zeta \otimes Q$ is injective, $\ker \rho_\zeta = \ker(\theta\rho)_\zeta$. On the other hand both $\ker \rho$ and $\ker \theta\rho$ are coherent sheaves, hence, after shrinking U we may assume (i) holds. We also note that

$$\begin{aligned}\dim \mathcal{C}_\zeta \otimes Q = \dim \mathcal{F}_\zeta \otimes Q = \dim \mathcal{N}_\zeta \otimes Q = \\ = \dim \operatorname{Im}(\rho^*_\zeta \otimes \mathrm{id}_Q) = \dim \operatorname{Im}(\rho_\zeta \otimes \mathrm{id}_Q).\end{aligned}$$

Consequently, the injectivity of the finite vector space homomorphism

$$\theta_\zeta \otimes \mathrm{id}_Q|_{\operatorname{Im}(\rho_\zeta \otimes \mathrm{id}_Q)} : \operatorname{Im}(\rho_\zeta \otimes \mathrm{id}_Q) \hookrightarrow \mathcal{C}_\zeta \otimes Q$$

implies its surjectivity, and (ii) follows immediately. □

Lemma 6.3. *Let $\mathcal{S}$ be a coherent sheaf over Ω. If $\zeta \in \Omega$, then there is a neighborhood $U \subset \Omega$ of ζ so that $\mathcal{S}(U) \to \mathcal{S}_\zeta$ is a monomorphism.*

Proof. We follow the outline given by the proof of Theorem 1.2. The lemma holds trivially for the sheaves over $\Omega = \mathbb{C}^0 = 0$. For Ω lying in higher dimensions we proceed by induction. Initially, we verify the inductive step for torsion modules; then, the general case is proved by reduction to a torsion case.

We are free to take $\zeta = 0$. As before Q denotes the field of quotients of $\mathcal{O}_0$. Suppose $\Omega \subset \mathbb{C}^m$, $m \geq 1$, and $\mathcal{S}_0 \otimes Q = 0$. Then $\mathcal{S}_0$ is annihilated by a nonzero

$h \in \mathcal{O}_0$. After choosing a suitable neighborhood $U = U' \times U'' \subset \mathbb{C}^{m-1} \times \mathbb{C}$ of 0 to be reduced further later, we can take h to be a Weierstrass polynomial of degree $d \geq 1$ in z_m and $h\mathcal{S}|_U = 0$. We write $z = (z', z_m)$ and $\mathcal{O}'$ for the sheaf of germs in $\mathbb{C}^{m-1}$.

Let $|A| = \{z \in U' \times U'' : h(z) = 0\}$, $|B| = U' \times \{0\}$, and $\mathcal{O}_A = (\mathcal{O}/h\mathcal{O})|_{|A|}$. Define complex spaces $A = (|A|, \mathcal{O}_A)$ and $B = (|B|, \mathcal{O}')$. Since $h\mathcal{S}|_U = 0$, $\operatorname{supp} \mathcal{S}|_U \subset |A|$ and $\mathcal{S}|_{|A|}$ has a structure of an $\mathcal{O}_A$-module. If $\mathcal{O}^r|_V \xrightarrow{\rho} \mathcal{O}^n|_V \xrightarrow{p} \mathcal{S}|_V \to 0$ is an exact sequence for some $V \subset U$, then the induced sequence $\mathcal{O}_A^r|_{|A|\cap V} \xrightarrow{\rho} \mathcal{O}_A^n|_{|A|\cap V} \xrightarrow{p} \mathcal{S}|_{|A|\cap V} \to 0$ is also exact. So, $\mathcal{S}|_{|A|}$ is also $\mathcal{O}_A$-coherent. The projection $U' \times U'' \to U'$ induces a holomorphic Weierstrass map $\pi : A \to B$, see [GR84, Section 2.3.4]. Since π is a finite map, the direct image sheaf $\pi_*(\mathcal{S}|_{|A|})$ is a coherent sheaf over $U' \subset \mathbb{C}^{m-1}$.

Inductively we can assume that U' is such that $\pi_*(\mathcal{S}|_{|A|})(U') \to \pi_*(\mathcal{S}|_{|A|})_0$ is a monomorphism. On the other hand

$$\pi_*(\mathcal{S}|_{|A|})(U') = \mathcal{S}|_{|A|}(\pi^{-1}U') = \mathcal{S}(U), \text{ and } \pi_*(\mathcal{S}|_{|A|})_0 = \prod_{\xi \in \pi^{-1}(0)} \mathcal{S}_\xi = \mathcal{S}_0,$$

see [GR84, Section 2.3.3]. So, $\mathcal{S}(U) \to \mathcal{S}_0$ is a monomorphism.

Now consider a general coherent sheaf $\mathcal{S}$. On some neighborhood U of ζ there exists an exact sequence $\mathcal{O}^r \xrightarrow{\rho} \mathcal{O}^n \xrightarrow{p} \mathcal{S}|_U \to 0$. This neighborhood U can be taken so that there exists a homomorphism $\theta : \mathcal{O}^n|_U \to \mathcal{O}^s|_U$ as in Lemma 6.2. Since $(\operatorname{coker} \theta\rho)_\zeta$ is a torsion $\mathcal{O}_\zeta$-module, we can apply the first part of the proof to assume that

$$(\operatorname{coker} \theta\rho)(U) \to (\operatorname{coker} \theta\rho)_\zeta \text{ is a monomorphism.} \tag{6.1}$$

Furthermore, we can take U to be a pseudoconvex domain.

Suppose $s \in \mathcal{S}(U)$ and $s_\zeta = 0$. Let $v \in \mathcal{O}^n(U)$ be such that $p(v) = s$. Then $v_\zeta \in \operatorname{Im} \rho_\zeta$ and the residue of $\theta_\zeta v_\zeta$ vanishes in $(\operatorname{coker} \theta\rho)_\zeta$. By (6.1), θv represents a zero section in $\operatorname{coker} \theta\rho|_U$, i.e., there is $w \in \mathcal{O}^r(U)$ with $\theta v = \theta\rho w$. Since $\ker \rho|_U = \ker \theta\rho|_U$, $v = \rho w$ and, thus, $s = 0$ □

Proof of Theorem 6.1. Let $\phi : \mathcal{O}^E \to \mathcal{S}$ be an $\mathcal{O}$-homomorphism. If $\zeta \in \Omega$, then according to Theorem 1.2 there is a plain homomorphism $\psi^\zeta : \mathcal{O}_\zeta^E \to \mathcal{O}_\zeta^n$ so that

$$\phi|_{\mathcal{S}_\zeta} = p\psi^\zeta. \tag{6.2}$$

Since ψ^ζ is induced by a homomorphism-valued holomorphic map, ψ^ζ extends to a plain homomorphism $\psi_U : \mathcal{O}^E|_U \to \mathcal{O}^n|_U$ for a neighborhood $U \subset \Omega$ of ζ. By Lemma 6.3, we can assume that $\mathcal{S}(U) \to \mathcal{S}_\zeta$ is a monomorphism. In conjunction with (6.2), this implies that $\phi(v) - p\psi_U(v) = 0$ for $v \in \mathcal{O}^E(U)$, in particular, for v a constant section. Then, an application of Lemma 3.4 shows that $\operatorname{Im}(\phi_{\zeta'} - (p\psi_U)_{\zeta'}) \subset \bigcap_{j=0}^\infty \mathfrak{m}_{\zeta'} \mathcal{S}_{\zeta'} = 0$ for $\zeta' \in U$, i.e., that ϕ factors through p on U. □

7. Applications

Our first application is Corollary 1.4. It depends on the following

Proposition 7.1. *Suppose $(R, \mathfrak{m})$ is a local ring with the residue field $k = R/\mathfrak{m}$, and M is a free R-module. If $c_\nu \in k$ for $\nu \in \mathbb{N}$ and $e_\nu \in M$ are such that their classes $\bar{e}_\nu$ in $M/\mathfrak{m}M$ are linearly independent over k then there is an R-homomorphism $\phi : M \to R$ such that for every ν the class of $\phi(e_\nu)$ in k is c_ν.*

Proof. Let $\bar{\phi} : M/\mathfrak{m}M \to k$ be a k-linear map such that $\bar{\phi}(\bar{e}_\nu) = c_\nu$. Composing $\bar{\phi}$ with the projection $M \to M/\mathfrak{m}M$ we obtain an R-homomorphism $\psi : M \to k$ such that $\psi(e_\nu) = c_\nu$. If M is free then ψ can be lifted to a $\phi : M \to R$ as required. $\square$

Proof of Corollary 1.4. Let $e_\nu \in \mathcal{O}^E_\zeta$ be germs such that $e_\nu(\zeta) \in E$ are $\mathbb{C}$-linearly independent unit vectors. Any $\mathcal{O}_\zeta$-homomorphism $\phi : \mathcal{O}^E_\zeta \to \mathcal{O}_\zeta$ is plain by Theorem 1.2, whence $\phi(e_\nu)(\zeta) \in \mathbb{C}$ is a bounded sequence. If $\mathcal{O}^E_\zeta$ were a submodule of a free module M then by Proposition 7.1 there would exist a homomorphism $\mathcal{O}^E_\zeta \to \mathcal{O}_\zeta$ such that $\phi(e_\nu)(\zeta) = \nu$, a contradiction. $\square$

For further applications we have to review some concepts introduced in [LP07]. As there, in this review we place ourselves in an open subset Ω of a Banach space X; but our applications will only concern finite dimensional X.

In the Introduction we have already defined plain sheaves and homomorphisms. For sheaves $\mathcal{A}, \mathcal{B}$ of $\mathcal{O}$-modules (always over Ω) we write $\mathbf{Hom}_\mathcal{O}(\mathcal{A}, \mathcal{B})$ for the sheaf of $\mathcal{O}$-homomorphisms between them; if $\mathcal{A}$ and $\mathcal{B}$ are plain sheaves we write $\mathbf{Hom}_{\text{plain}}(\mathcal{A}, \mathcal{B}) \subset \mathbf{Hom}_\mathcal{O}(\mathcal{A}, \mathcal{B})$ for the sheaf of plain homomorphisms.

Definition 7.2. An analytic structure on a sheaf $\mathcal{S}$ is the choice, for each plain sheaf $\mathcal{E}$, of a submodule $\mathbf{Hom}(\mathcal{E}, \mathcal{S}) \subset \mathbf{Hom}_\mathcal{O}(\mathcal{E}, \mathcal{S})$ subject to

- If $\mathcal{E}, \mathcal{F}$ are plain sheaves, $x \in \Omega$, and $\varphi \in \mathbf{Hom}_{\text{plain}}(\mathcal{E}, \mathcal{F})_x$, then $\varphi^* \mathbf{Hom}(\mathcal{F}, \mathcal{S})_x \subset \mathbf{Hom}(\mathcal{E}, \mathcal{S})_x$; and
- $\mathbf{Hom}(\mathcal{O}, \mathcal{S}) = \mathbf{Hom}_\mathcal{O}(\mathcal{O}, \mathcal{S})$.

If $\mathcal{S}$ is endowed with an analytic structure, one also says that $\mathcal{S}$ is an analytic sheaf. This terminology is different from the traditional one, where "analytic sheaves" and "sheaves of $\mathcal{O}$-modules" mean one and the same thing.

If $U \subset \Omega$ is open, an $\mathcal{O}$-homomorphism $\psi : \mathcal{S}|_U \to \mathcal{S}'|_U$ of analytic sheaves is called analytic if $\psi_* \mathbf{Hom}(\mathcal{E}|_U, \mathcal{S}|_U) \subset \mathbf{Hom}(\mathcal{E}|_U, \mathcal{S}'|_U)$ for every plain sheaf $\mathcal{E}$.

Any plain sheaf $\mathcal{F}$ has a canonical analytic structure given by $\mathbf{Hom}(\mathcal{E}, \mathcal{F}) = \mathbf{Hom}_{\text{plain}}(\mathcal{E}, \mathcal{F})$. Further, on any $\mathcal{O}$-module $\mathcal{S}$ one can define a "maximal" analytic structure by $\mathbf{Hom}(\mathcal{E}, \mathcal{S}) = \mathbf{Hom}_\mathcal{O}(\mathcal{E}, \mathcal{S})$; and also a "minimal" analytic structure, denoted by $\mathbf{Hom}_{\min}(\mathcal{E}, \mathcal{S})$, consisting of germs α that can be written as a composition $\beta\gamma$ of

$$\gamma \in \mathbf{Hom}_{\text{plain}}(\mathcal{E}, \mathcal{O}^n) \text{ and } \beta \in \mathbf{Hom}_\mathcal{O}(\mathcal{O}^n, \mathcal{S}),$$

where $n < \infty$. Definition 7.2 implies that

$$\mathbf{Hom}_{\min}(\mathcal{E}, \mathcal{S}) \subset \mathbf{Hom}(\mathcal{E}, \mathcal{S}) \subset \mathbf{Hom}_\mathcal{O}(\mathcal{E}, \mathcal{S}).$$

In view of Theorems 1.1 and 6.1 we obtain the following uniqueness results

Theorem 7.3. *For every plain sheaf $\mathcal{O}^F$ over an open $\Omega \subset \mathbb{C}^m$, $0 < m < \infty$, the canonical and the maximal analytic structures coincide.*

Proof. Let E be a Banach space, $U \subset \Omega$ an open set, and $\phi : \mathcal{O}^E|_U \to \mathcal{O}^F|_U$ an $\mathcal{O}$-homomorphism. By Theorem 1.1, ϕ is a plain homomorphism, and hence, an analytic homomorphism for the canonical analytic structure. Thus, $\mathbf{Hom}(\mathcal{O}^E, \mathcal{O}^F) = \mathbf{Hom}_\mathcal{O}(\mathcal{O}^E, \mathcal{O}^F)$. $\square$

Theorem 7.4. *Let $\mathcal{S}$ be a coherent sheaf such that* $\operatorname{depth}_{\mathfrak{m}_\zeta} \mathcal{S}_\zeta > 0$ *for* $\zeta \in \operatorname{supp} \mathcal{S}$. *Then the minimal and the maximal analytic structures coincide, i.e., $\mathcal{S}$ has unique analytic structure.*

Proof. Denote by $\Omega \subset \mathbb{C}^m$ the base of the sheaf $\mathcal{S}$. Let E be a Banach space, $U \subset \Omega$ an open set, and $\phi : \mathcal{O}^E|_U \to \mathcal{O}^F|_U$ an $\mathcal{O}$-homomorphism. If $m = 0$, the depth condition guarantees that $\mathcal{S} = 0$ and the conclusion of the theorem follows. So, we may assume that $m \geq 1$.

Since $\mathcal{S}$ is a coherent sheaf, given $\zeta \in U$ there is an epimorphism $p : \mathcal{O}^n|_V \to \mathcal{S}|_V$, with $n < \infty$ and $V \subset U$, a suitable neighborhood of ζ. By Theorem 6.1, we can assume that $\phi|_V$ factors through $p|_V$, i.e., there is an $\mathcal{O}$-homomorphism $\psi : \mathcal{O}^E|_V \to \mathcal{O}^n|_V$ with $\phi|_V = p|_V\psi$. Then, by Theorem 1.1, ψ is a plain homomorphism, and so, $\phi_\zeta \in \mathbf{Hom}_{\min}(\mathcal{O}^E, \mathcal{O}^F)_\zeta$. Since ϕ and ζ were arbitrary, it follows that $\mathbf{Hom}_{\min}(\mathcal{O}^E, \mathcal{O}^F) = \mathbf{Hom}_{\mathcal{O}}(\mathcal{O}^E, \mathcal{O}^F)$. □

8. Epimorphisms on Coherent Sheaves

Theorem 8.1. *Let $\mathcal{S}$ be a coherent sheaf over an open pseudoconvex $\Omega \subset \mathbb{C}^m$, and E be a Banach space. Suppose* $\operatorname{depth}_{\mathfrak{m}_\zeta} \mathcal{S}_\zeta > 0$ *for $\zeta \in \operatorname{supp} \mathcal{S}$. If $p : \mathcal{O}^n \to \mathcal{S}$ is an epimorphism, then any $\mathcal{O}$-homomorphism $\mathcal{O}^E \to \mathcal{S}$ factors through it.*

Proof of Theorem 8.1. This proof is based on the following key fact [Lem, Theorem 4.3] due to Lempert: a coherent sheaf over $\Omega \subset \mathbb{C}^m$ endowed with its minimal analytic structure is cohesive. While [Lem] makes references to some of the results of the present paper, the proof of [Lem, Theorem 4.3] is independent of Theorem 8.1.

Let $\phi : \mathcal{O}^E \to \mathcal{S}$ be an $\mathcal{O}$-homomorphism. In view of Theorem 6.1, there is an open pseudoconvex cover $\mathfrak{V}$ of Ω such that on each $V \in \mathfrak{V}$ there is a homomorphism $\psi_V : \mathcal{O}^E|_V \to \mathcal{O}^n|_V$ with $\phi|_V = p\psi|_V$. If we let $\mathcal{K} = \ker p$, a coherent sheaf, then $\psi_{VW} = \psi_V - \psi_W$ maps $\mathcal{O}^E_{V\cap W}$ into $\mathcal{K}$, for $V, W \in \mathfrak{V}$. Thus, the $\mathcal{O}$-homomorphisms ψ_{VW} form a $\mathcal{K}$-valued 1-cocycle.

We can assume that $m \geq 1$, for otherwise, the depth condition implies that $\mathcal{S}$ is a zero-sheaf and there is nothing to prove. The module $\mathcal{K} \subset \mathcal{O}^n$ is torsion-free, i.e., $r_\zeta k_\zeta \neq 0$ for $\zeta \in \Omega$, $r_\zeta \in \mathcal{O}_\zeta$, and $k_\zeta \in \mathcal{K}_\zeta$, unless $r_\zeta = 0$ or $k_\zeta = 0$. Therefore, in view of Lemma 4.2, $\operatorname{depth}_{\mathfrak{m}_\zeta} \mathcal{K}_\zeta > 0$ for all $\zeta \in \operatorname{supp} \mathcal{K}$. We endow $\mathcal{K}$ with the minimal analytic structure, and note that, by Theorem 7.4, ψ_{VW} are analytic with respect to this structure. On the other hand, $\mathcal{K}$ is coherent, and hence, by [Lem, Theorem 4.3], is cohesive. Now $H^1(\Omega, \mathbf{Hom}(\mathcal{O}^E, \mathcal{K})) = 0$, which is a special case of [LP07, Theorem 9.1]. Consequently, $\psi_{VW} = \theta_V - \theta_W$ with some (analytic) homomorphisms $\theta_V : \mathcal{O}^E|_V \to \mathcal{K}|_V$; defining $\psi : \mathcal{O}^E \to \mathcal{O}^n$ by

$$\psi|_V = \psi_V - \theta_V,$$

the resulting homomorphism satisfies $\phi = p\psi$. □

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